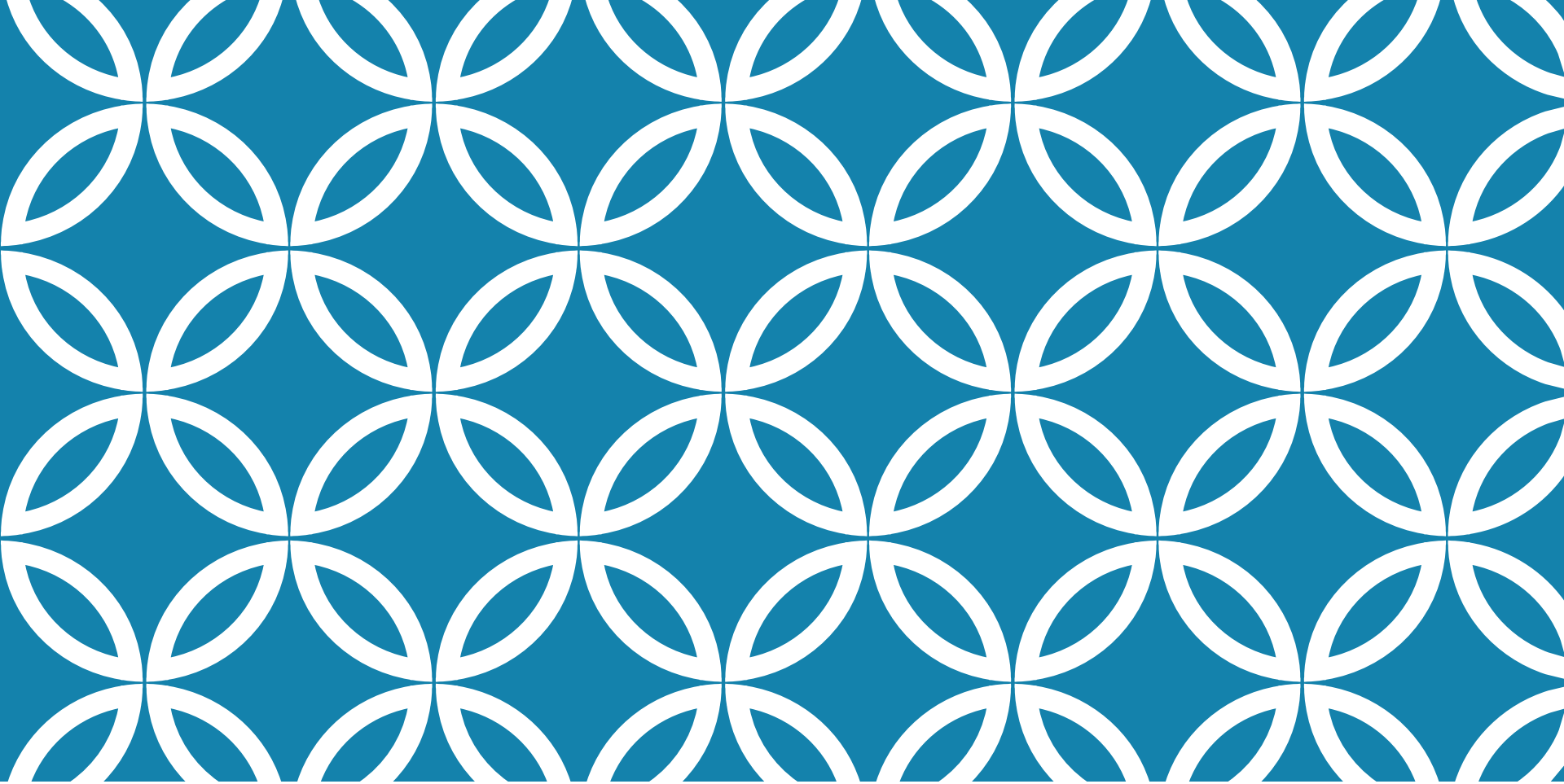


CONTINUOUS PROBABILITY DISTRIBUTIONS

Chapter 7

CONTINUOUS PROBABILITY DISTRIBUTIONS

A fundamental difference separates discrete and continuous random variables in terms of how probabilities are computed. For a discrete random variable, the probability function p(x) provides the probability that the random variable assumes a particular value. with continuous random variables. The counterpart of the probability function is the Probability Density Function. Also denoted by f(x).

The difference is that the probability density function does not directly provide probabilities. However, the area under the graph of f(x) corresponding to a given interval does provide the probability that the continuous random variable x assumes a value in that interval.

So when we compute probabilities for continuous random variables we are computing the probability that the random variable assumes any value in an interval

EXAMPLE 01

Note : Because the area under the graph of f(x) at any particular point is 0, one of the implications of the definition of probability for continuous random variables is that the probability of any particular value of then random variable is 0.

Example: Consider the function $f(x) = \begin{cases} k - \frac{x}{4} & \dots\dots 1 \leq x \leq 3 \\ 0, elsewhere \end{cases}$

which is being used as a probability density function for a continuous random variable x.

1. Find the value of k ?
2. Find $p(x \leq 2.5)$?

SOLUTION 01

1. By definition the total area under the curve of probability density function = 1.

$$\int_1^3 f(x).dx = 1 \Rightarrow \int_1^3 f(x).dx = 1 \Rightarrow \int_1^3 k - \frac{x}{4}.dx = 1 \Rightarrow \left[kx - \frac{x^2}{2*4}\right]_1^3 = 1$$

$$\left[kx - \frac{x^2}{8}\right]_1^3 = 1 \Rightarrow \left[3k - \frac{9}{8}\right] - \left[k - \frac{1}{8}\right] = 1$$

$$3k - \frac{9}{8} - k + \frac{1}{8} = 1$$

$$2k = \left(\frac{9}{8} - \frac{1}{8}\right) + 1 \Rightarrow 2k = 2 \Rightarrow k = 1$$

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EXAMPLE (CONT).

thus:

$$f(x) = \begin{cases} 1 - \frac{x}{4} & \dots\dots 1 \le x \le 3 \\ 0, elsewhere \end{cases}$$

$$\mathrm{p}(\mathrm{x} \le 2{,}5) = \int_1^{2.5} 1 - \frac{\mathrm{x}}{4} . \mathrm{dx} = \left[x - \frac{x^2}{8} \right]_1^{2.5}$$

$$\left[2.5 - \frac{2.5^2}{8} \right] - \left[1 - \frac{1^2}{8} \right] = \frac{13.75}{8} - \frac{7}{8} = 0.8437$$

EXPECTED VALUE AND VARIANCE OF CONTINUOUS RANDOM VARIABLE

$$\mathbf{E(x) = \mu = \int_{-\infty}^{+\infty} x * f(x)}$$

$$\mathbf{Var(x) = m_2 - m_1 = E\left(x^2\right) - E(x)^2}$$

$$\mathbf{= \int_{-\infty}^{+\infty} x^2 * f(x)dx - (m_1)^2}$$

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EXAMPLE 02

The random variable x has probability density function given by:

$$f(x) = \begin{cases} \frac{3}{10}(3x - x^2) \ \ldots 0 \leq x \leq 2 \\ 0 \ \ elsewhere \end{cases}$$

- Find the mean and variance of x ?

SOLUTION 02

- $E(x) = \int_0^2 xf(x).dx = \int_0^2 x\frac{3}{10}(3x - x^2)$

$$= \int_0^2 \left(\frac{9}{10}x^2 - \frac{3}{10}x^3\right)$$

$$= \left[\frac{9}{30}x^3 - \frac{3}{40}x^4\right]_0^2$$

$$= \left[\frac{9}{30}2^3 - \frac{3}{40}2^4\right] - \left[\frac{9}{30}0^3 - \frac{3}{40}0^4\right]$$

$$= \left[\frac{72}{30} - \frac{48}{40}\right] - 0 = \frac{144}{120} = \frac{6}{5}$$

EXAMPLE 02 (CONT.)

- $\mathrm{Var}(x) = \mathrm{E}(x^2) - \mu^2 = \int_0^2 x^2 f(x).\,dx - \mu^2$
- $\mathrm{E}(x^2) = \int_0^2 x^2 \frac{3}{10}(3x - x^2)\,.\,dx = \int_0^2 \frac{9}{10}x^3 - \frac{3}{10}x^4.\,dx$
- $= \left[\frac{9}{40}x^4 - \frac{3}{50}x^5\right]_0^2 = \left[\frac{9}{40}2^4 - \frac{3}{50}2^5\right] - \left[\frac{9}{40}0^4 - \frac{3}{50}0^5\right]$
- $\left[\frac{144}{40} - \frac{96}{50}\right] - 0 = \frac{42}{25}$
- $Var(x) = \frac{42}{25} - \left(\frac{6}{5}\right)^2 = \frac{6}{25} = 0{,}24$

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NORMAL PROBABILITY DISTRIBUTION

The most commonly used probability distribution for describing a continuous random variable is the **normal probability distribution.** The normal distribution has been used in a wide variety of practical applications in which the random variables are heights and weights of people, test scores, scientific measurements, amounts of rainfall, and other similar values.

The normal probability distribution is a continuous distribution with the following characteristics:

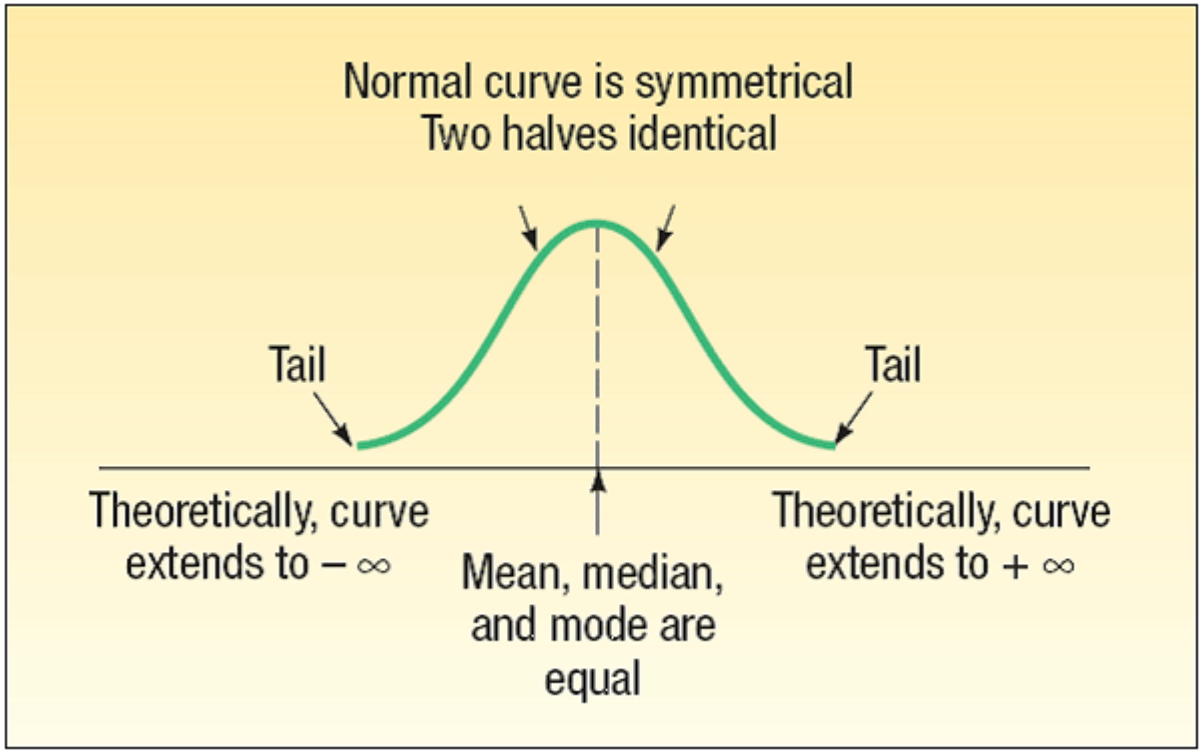
NORMAL PROBABILITY DISTRIBUTION

- It is bell-shaped and has a single peak at the center of the distribution
- The distribution is symmetric
- It is asymptotic, meaning the curve approaches but never touches the X-axis
- It is completely described by its mean and standard deviation

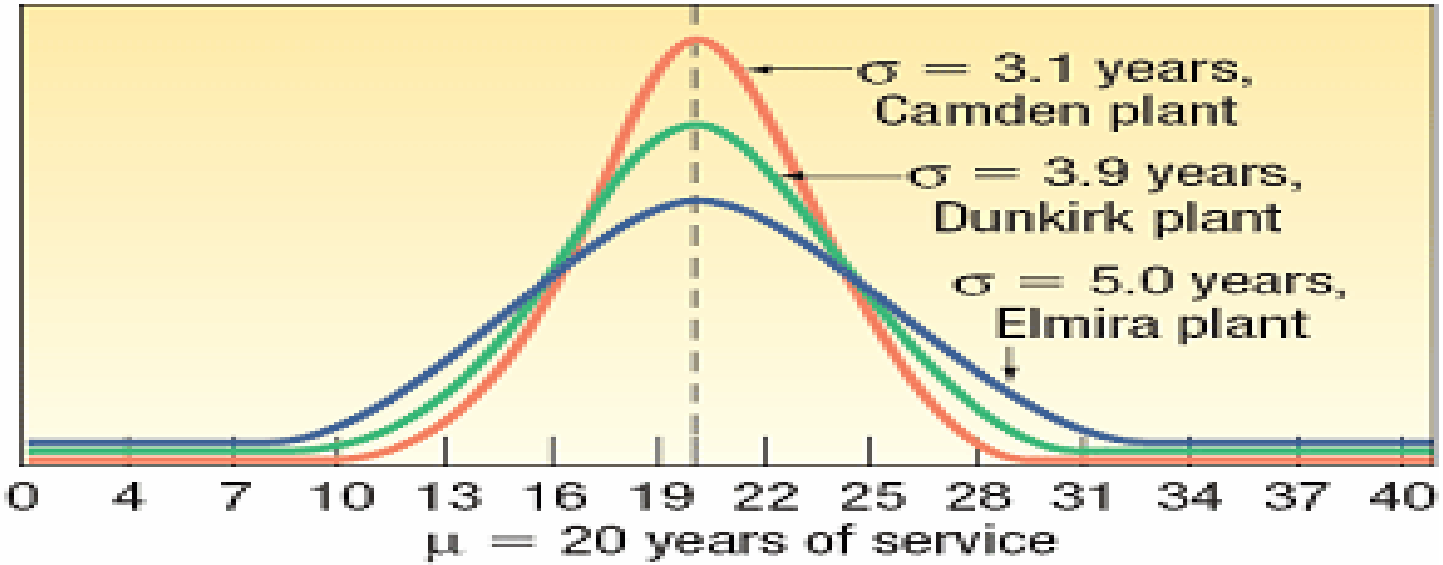
There is a family of normal probability distributions

- Another normal probability distribution is created when either the mean or the standard deviation changes

THE NORMAL CURVE

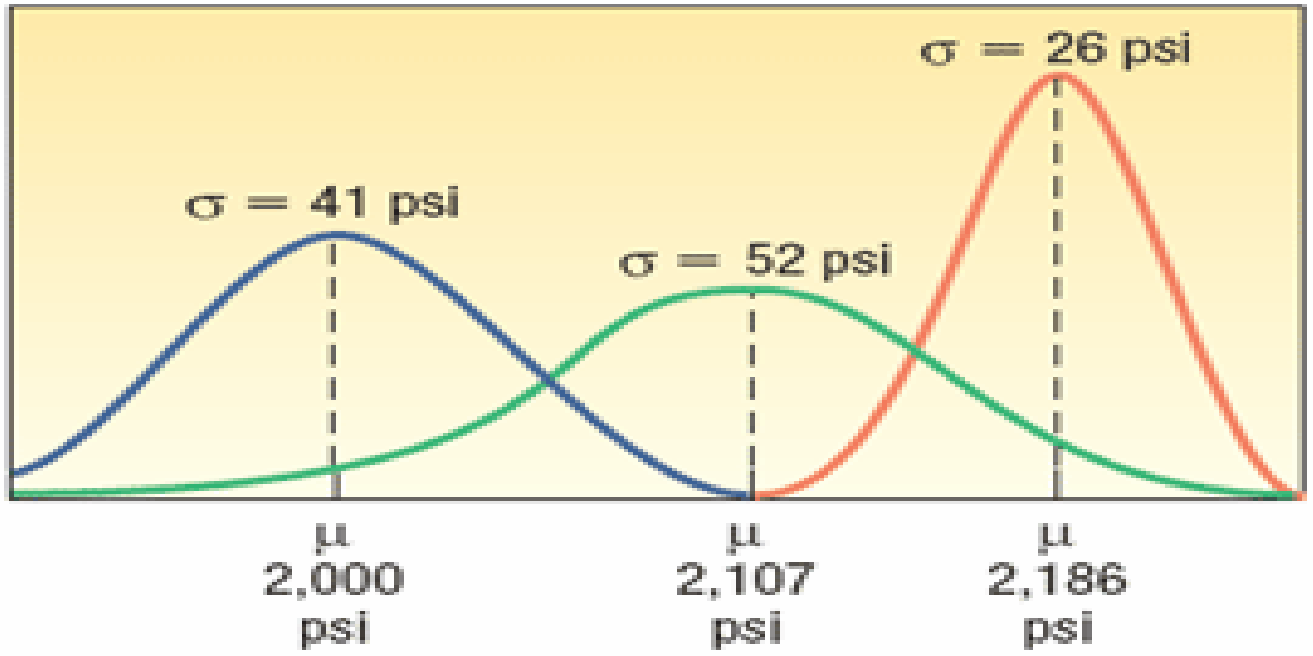


FAMILY OF NORMAL PROBABILITY DISTRIBUTIONS



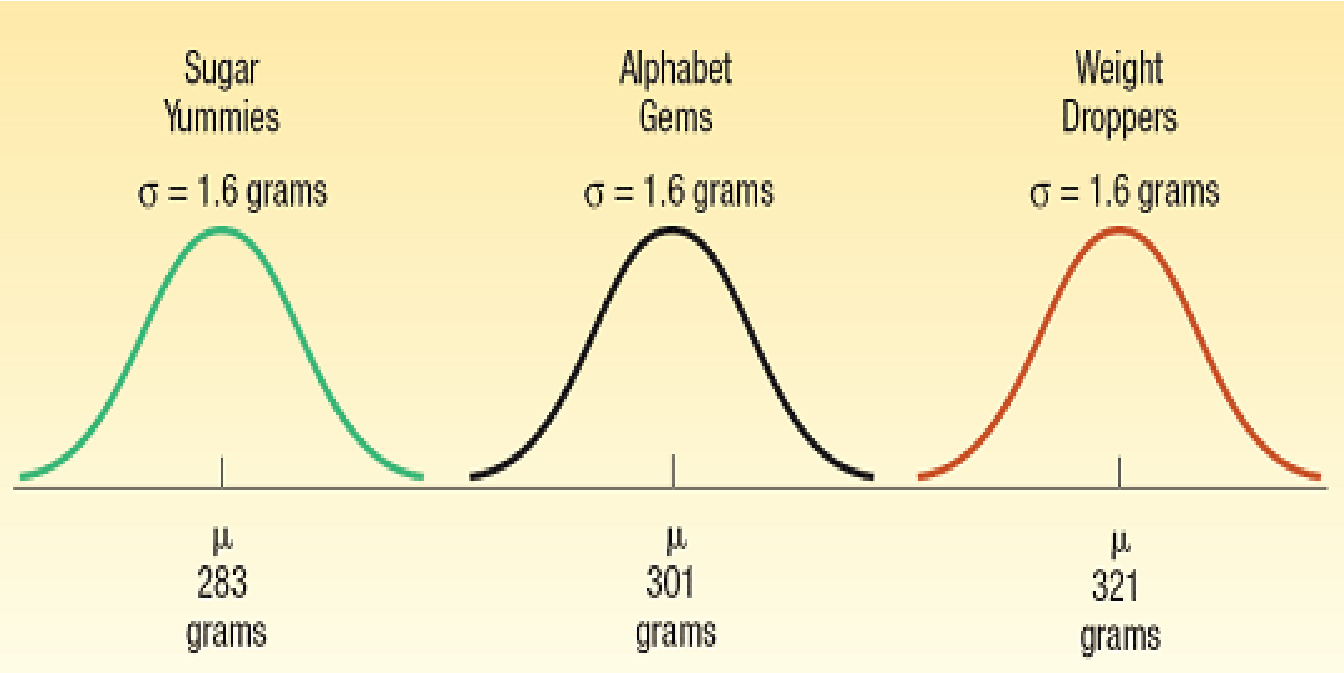
Equal Means and Different Standard Deviations

FAMILY OF NORMAL PROBABILITY DISTRIBUTIONS (CONT.)



Different Means and Standard Deviations

FAMILY OF NORMAL PROBABILITY DISTRIBUTIONS (CONT.)



Different Means and Equal Standard Deviations

NORMAL PROBABILITY DENSITY FUNCTION

A normal curve with mean μ and standard deviation σ can be graphed using the normal probability density function

Normal Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Where: $\mu = \text{mean}$

$\sigma = \text{standard deviation}$

$\pi = 3.14159$

$e = 2.71828$

NORMAL PROBABILITY DISTRIBUTION PROPERTIES

There are several properties of normal distribution:

a. The entire family of the normal distribution is differentiated by two parameters: the mean μ and the standard deviation σ.

b. The mean, median and mode are equal.

c. The highest point on the normal curve is at the mean, which is also the median and mode of the distribution.

d. The normal curve is bell-shaped and is symmetric about the mean.

NORMAL PROBABILITY DISTRIBUTION PROPERTIES (CONT.)

e. The standard deviation determines how flat and wide the normal curve is. larger values of the standard deviation result in wider, flatter curves, showing more variability in the data.

f. The total area under the curve is equal to 1.

g. The normal curve approaches but never touches the x axis as it extends farther and farther away from the mean.

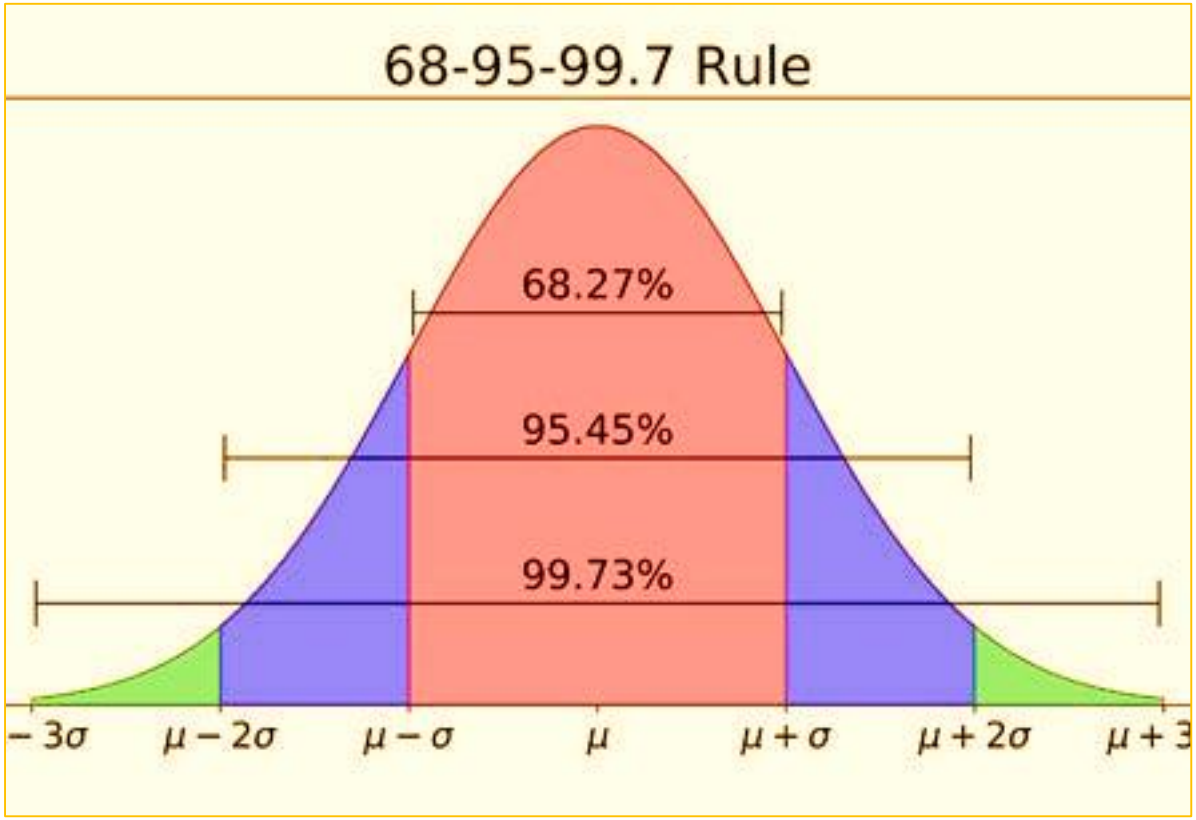
h. Because the distribution is symmetric, the area under the curve to the left of the mean is 0.5 and the area under the curve to the right of the mean is 0.5.

NOTE:

Another property of Normal distribution is that:

- 68.27% of the values of normal random variable are within plus or minus one standard deviation of its mean.
- 95.45% of the values of a normal random variable are within plus or minus two standard deviations of its mean.
- 99.73% of the values of a normal random variable are within plus minus three standard deviations of its mean

NOTE:(CONT.)



STANDARD NORMAL PROBABILITY DISTRIBUTION

The standard normal probability distribution is a particular normal distribution

- It has a mean of 0 and a standard deviation of 1

z VALUE The signed distance between a selected value, designated x, and the mean, μ, divided by the standard deviation, σ.

Any normal probability distribution can be converted to the standard normal probability distribution with the following formula:

$$Z = \frac{X - \mu}{\sigma}$$

STANDARD NORMAL PROBABILITY DISTRIBUTION (DENSITY FUNCTION)

Standard Normal Density Function:

Because $\mu = 0$ and $\sigma = 1$

Standard Normal Density Function:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

(CONT.)

to find the probability that a normal random variable is within any specific interval, we must compute the area under the normal curve over that interval.

For the standard normal distribution, areas under the normal curve have been computed and are available tables that can be used to compute probabilities.

The table contains cumulative probabilities or areas for z values greater than or equal to the mean of zero.

(CONT.)

Types of probabilities for standard normal probability distribution:

Three types of probabilities we need to compute include:

1. the probability that the standard normal random variable z will be less than or equal to a given value.
2. The probability that z will be between two given values.
3. The probability that z will be greater than or equal to a given value.

EXEMPLES FOR THREE TYPES OF PROBABILITIES

1. **The probability that the standard normal random variable z will be less than or equal to a given value.**

Example:

Find the probability or the cumulative area for z less than 1.15, that is p ($z \leq 1.15$).

Find the probability or the cumulative area for z less than -0.24, that is $p\ (z \leq -0.24)$

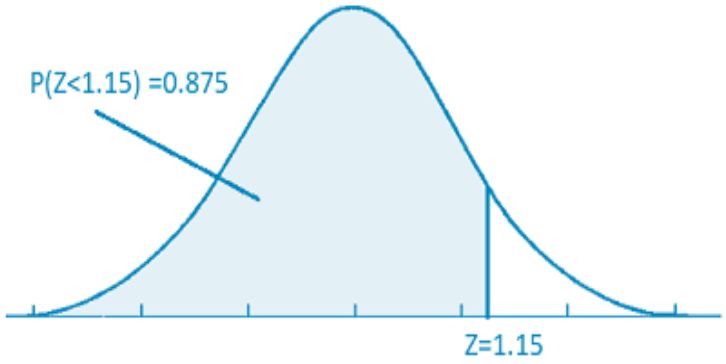
Solution:

Refer to the standard normal probability table.

(CONT.)

First, we need to find the area that corresponds to z = 1.15. By finding the 1.1 in the left column and then moving across the row to the column under 0.05, by looking in the body of the table we find that the 1.1 row and the 0.05 column intersect at the value 0.8749.

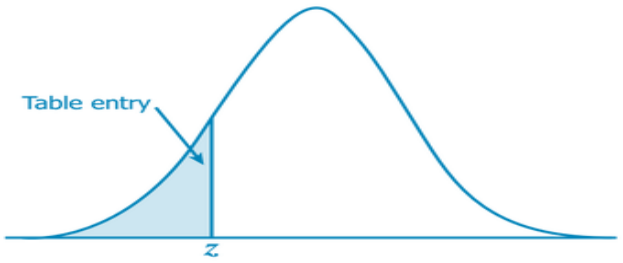
Thus, $(z \leq 1.15) = 0.8749$.



(CONT.)

pp ($z \leq -0.24$) The table we are using gives only cumulative probabilities for z values greater than or equal to the mean of zero (positive values of z). Since the standard normal probability distribution is symmetric distribution, thus, p ($z \leq -0.24$)=$p(z \geq 0.24)$.

The standard normal distribution also does not provide us with probability of z values greater than or equal a given value, but still give us the probability of $p(z \leq 0.24)$.And because the total area under the normal curve is 1.Thus, p ($z \leq -0.24$)=$p(z \geq 0.24)$=1−$p(z \leq 0.24)$.=1−0.5948=0.405



(CONT.)

2. **The probability that z will be between two given values.**

Find the probability or the cumulative area for z between -1.00 and 1.00

that is $p(-1 \leq z \leq 1)$.

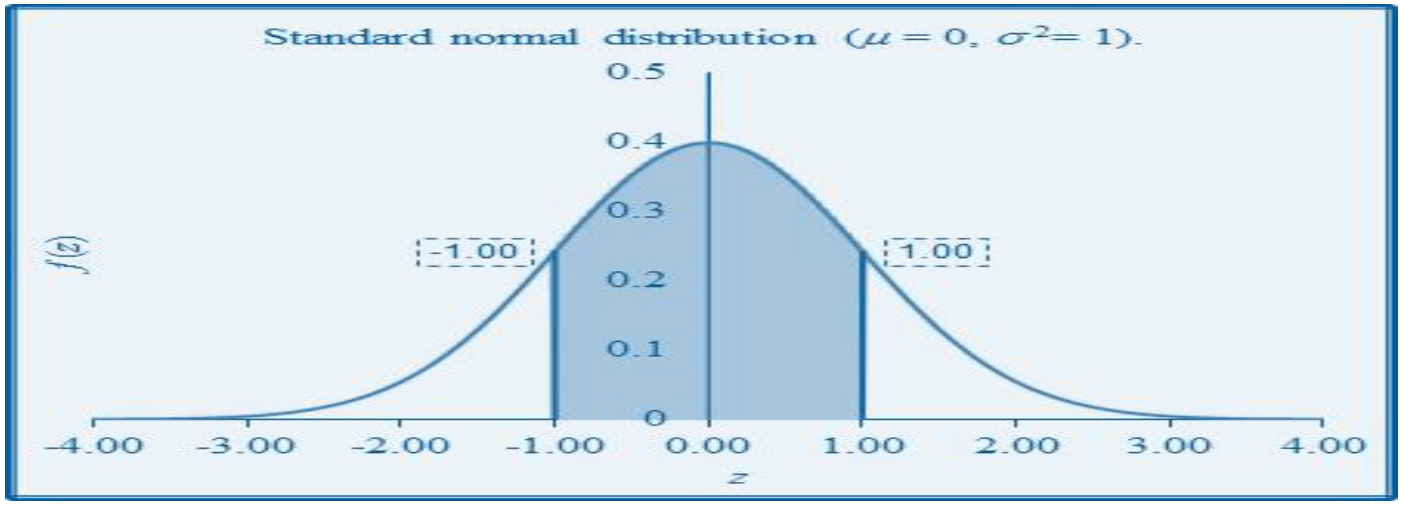
Solution:

First, we need to find $p(z \leq -1) = (z \geq 1) = 1 - p(z \leq 1) = 1 - 0.8413 = 0.1587$

$pp(z \leq 1) = 0.8413$

$pp(-1 \leq z \leq 1) = p(z \leq 1) - p(z \leq -1) = 0.8413 - 0.1587 = \mathbf{0.6826}$

(CONT.)



3. **The probability that z will be greater than or equal to a given value**

Find the probability or the area of $z \geq 1.06$, that is $p(z \geq 1.06)$.

$p(z \geq 1.06) = 1 - p(z \leq 1.06) = 1 - 0.8554 = 0.1446$